

CONCAVE-CONVEX NONLINEARITIES FOR SOME NONLINEAR FRACTIONAL EQUATIONS INVOLVING THE BESSEL OPERATOR

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ABSTRACT. We prove some existence results for a class of nonlinear fractional equations driven by a nonlocal operator.

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1. INTRODUCTION

In this paper we provide some existence results for a class of nonlinear fractional equations of the form

$$(1.1) \quad (I - \Delta)^\alpha u + \lambda V(x)u = f(x, u) + \mu \xi(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N$$

where $0 < \alpha < 1$, $V: \mathbb{R}^N \rightarrow \mathbb{R}$, $f: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, $\xi: \mathbb{R}^N \rightarrow (0, +\infty)$ belongs to $L^{2/(2-p)}(\mathbb{R}^N)$, $\lambda > 0$, $\mu > 0$ and $1 < p < 2$.

The operator $(I - \Delta)^\alpha$ is related to the pseudo-relativistic Schrödinger operator $(m^2 - \Delta)^{1/2} - m$ ($m > 0$) and recently a lot of attention is paid to equations involving it. We refer to [3–5, 12–15, 17, 19, 26, 30] and references therein for more details and physical context of $(1 - \Delta)^\alpha$. In these papers, the authors study the existence of nontrivial solution and infinitely many solutions for the equations with $(m^2 I - \Delta)^\alpha$ and various nonlinearities. We remark that the hardest issue in dealing with this operator is the lack of scaling properties: there is no standard group action under which $(I - \Delta)^\alpha$ behaves as a local differential operator.

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In particular, Felmer *et al.* proved in [19] prove the existence of positive solution of $(I - \Delta)^\alpha u = f(x, u)$ under the following conditions on $f(x, s)$:

- (i) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.
- (ii) For all $x \in \mathbb{R}^N$, $f(x, s) \geq 0$ if $s \geq 0$ and $f(x, s) = 0$ if $s \leq 0$.
- (iii) The function $s \mapsto s^{-1}f(x, s)$ is increasing in $(0, \infty)$ for all $x \in \mathbb{R}^N$.
- (iv) There are $1 < p < 2_\alpha^* - 1 = (N + 2\alpha)/(N - 2\alpha)$ and $C > 0$ such that $|f(x, s)| \leq C|s|^p$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$.
- (v) There exists a $\mu > 2$ such that $0 < \mu F(x, s) \leq sf(x, s)$ for all $(x, s) \in \mathbb{R}^N \times (0, \infty)$ where $F(x, s) := \int_0^s f(x, t) dt$.
- (vi) There exist continuous functions $\bar{f}(s)$ and $a(x)$ such that \bar{f} satisfies (i)–(v) and $0 \leq f(x, s) - \bar{f}(s) \leq a(x)(|s| + |s|^p)$ for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}$, $a(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and

$$\mathcal{L}^N(\{x \in \mathbb{R}^N \mid f(x, s) > \bar{f}(s) \text{ for all } s > 0\}) > 0.$$

Earlier this year, Ikoma proved in [21] the existence of a solution to the equation $(I - \Delta)^\alpha u = f(u)$ under the so-called Berestycki-Lions assumptions on the nonlinearity f . As a consequence, Ikoma proved also the existence of a solution to the non-autonomous equation $(I - \Delta)^\alpha u = f(x, u)$ with $f(x, s) = -V(x)s + g(x, s)$ and suitable assumptions on V and g .

We continue the investigation initiated in [26], and we present two existence results. We fix our standing assumptions on the nonlinear term f :

- (f1) $|f(x, u)| \leq c(1 + |u|^{q-1})$ for some $q \in (2, 2_\alpha^*)$, where $2_\alpha^* = 2N/(N - 2\alpha)$;
- (f2) $f(x, u) = o(|u|)$ as $u \rightarrow 0$ uniformly with respect to $x \in \mathbb{R}^N$;
- (f3) there exists a constant $\vartheta > 2$ such that $0 < \vartheta F(x, u) \leq uf(x, u)$ for every $x \in \mathbb{R}^N$ and $u \neq 0$, where $F(x, u) = \int_0^u f(x, s) ds$.

The first, Theorem 3.7, shows that (1.1) possesses, for every $\lambda > 0$ and $\mu > 0$, at least a nontrivial solution if V has some coercivity property. The second, Theorem 4.1, shows that (1.1) possesses at least two nontrivial solutions if $\lambda > 0$ is sufficiently large and $\mu > 0$ is sufficiently small, but without any coercivity on the potential V . Our present assumptions are weaker than those in [26], and more general models are allowed.

In both cases we use variational methods to perform our analysis. The fact that (1.1) is set in the whole \mathbb{R}^N introduces a natural lack of compactness that must be overcome before applying Critical Point Theory.

Notation

- (1) The letters c and C will stand for a generic positive constant that may vary from line to line.
- (2) The operator D will be reserved for the (Fréchet) derivative.
- (3) The symbol \mathcal{L}^N will be reserved for the Lebesgue N -dimensional measure.
- (4) The Fourier transform of a function f will be denoted by $\mathcal{F}u$.
- (5) For a real-valued function V , we use the notation $V^b = \{x \in \mathbb{R}^N \mid V(x) < b\}$.

2. PRELIMINARIES AND FUNCTIONAL SETTING

For $\alpha > 0$ we introduce the *Bessel function space*

$$L^{\alpha,2}(\mathbb{R}^N) = \{f : f = G_\alpha \star g \text{ for some } g \in L^2(\mathbb{R}^N)\},$$

where the Bessel convolution kernel is defined by

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2} \Gamma(\alpha/2)} \int_0^\infty \exp\left(-\frac{\pi}{t}|x|^2\right) \exp\left(-\frac{t}{4\pi}\right) t^{\frac{\alpha-N}{2}-1} dt$$

The norm of this Bessel space is $\|f\| = \|g\|_2$ if $f = G_\alpha \star g$. The operator $(I - \Delta)^{-\alpha} u = G_{2\alpha} \star u$ is usually called Bessel operator of order α .

In Fourier variables the same operator reads

$$G_\alpha = \mathcal{F}^{-1} \circ \left((1 + |\xi|^2)^{-\alpha/2} \circ \mathcal{F} \right),$$

so that

$$\|f\| = \left\| (I - \Delta)^{\alpha/2} f \right\|_2.$$

For more detailed information, see [1, 27] and the references therein.

In the paper [17] the pointwise formula

$$(I - \Delta)^\alpha u(x) = c_{N,\alpha} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{\frac{N+2\alpha}{2}}} K_{\frac{N+2\alpha}{2}}(|x - y|) dy + u(x)$$

was derived for functions $u \in C_c^2(\mathbb{R}^N)$. Here $c_{N,\alpha}$ is a positive constant depending only on N and α , P.V. denotes the principal value of the singular integral, and K_ν is the modified Bessel function of the second kind with order ν (see [17, Remark 7.3] for more details). However a closed formula for K_ν is not known.

We summarize the embedding properties of Bessel spaces. For the proofs we refer to [18, Theorem 3.1], [27, Chapter V, Section 3] and [29, Section 4].

- Theorem 2.1.** (1) $L^{\alpha,2}(\mathbb{R}^N) = W^{\alpha,2}(\mathbb{R}^N) = H^\alpha(\mathbb{R}^N)$.
 (2) If $\alpha \geq 0$ and $2 \leq q \leq 2_\alpha^* = 2N/(N-2\alpha)$, then $L^{\alpha,2}(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$; if $2 \leq q < 2_\alpha^*$ then the embedding is locally compact.
 (3) Assume that $0 \leq \alpha \leq 2$ and $\alpha > N/2$. If $\alpha - N/2 > 1$ and $0 < \mu \leq \alpha - N/2 - 1$, then $L^{\alpha,2}(\mathbb{R}^N)$ is continuously embedded into $C^{1,\mu}(\mathbb{R}^N)$. If $\alpha - N/2 < 1$ and $0 < \mu \leq \alpha - N/2$, then $L^{\alpha,2}(\mathbb{R}^N)$ is continuously embedded into $C^{0,\mu}(\mathbb{R}^N)$.

Remark 2.2. Although the Bessel space $L^{\alpha,2}(\mathbb{R}^N)$ is topologically undistinguishable from the Sobolev fractional space $H^\alpha(\mathbb{R}^N)$, we will not confuse them, since our equation involves the Bessel norm.

The next result will be useful in the sequel. We define

$$\mathcal{F}(x, u) = \frac{1}{2}f(x, u) - F(x, u)$$

for $(x, u) \in \mathbb{R} \times \mathbb{R}$.

Lemma 2.3. For every $\tau \in \left(\max\left\{1, \frac{N}{2\alpha}\right\}, \frac{q}{q-2}\right)$ there exists $R > 0$ such that $|u| \geq R$ implies

$$\frac{|f(x, u)|^\tau}{|u|^\tau} \leq \mathcal{F}(x, u).$$

Proof. It follows from (f1)–(f3) that $|f(x, u)| \leq C|u|^{q-1}$ if $|u|$ is large enough. Now,

$$F(x, u) \leq \frac{1}{\vartheta}uf(x, u) + \frac{1}{2}uf(x, u) - \frac{1}{2}uf(x, u),$$

or

$$\left(\frac{1}{2} - \frac{1}{\vartheta}\right)uf(x, u) \leq \mathcal{F}(x, u).$$

We claim that, for $|u|$ large enough,

$$\frac{|f(x, u)|^\tau}{|u|^\tau} \leq \left(\frac{1}{2} - \frac{1}{\vartheta}\right)uf(x, u).$$

It suffices to prove that

$$\frac{|f(x, u)|^{\tau-1}}{|u|^{\tau+1}} \leq \frac{1}{2} - \frac{1}{\vartheta};$$

recalling that $|f(x, u)|^{\tau-1} \leq C^{\tau-1}|u|^{(q-1)(\tau-1)}$ for $|u|$ large enough, we deduce that

$$\frac{|f(x, u)|^{\tau-1}}{|u|^{\tau+1}} \leq C^{\tau-1} \frac{|u|^{(\tau-1)(q-1)}}{|u|^{\tau+1}} \leq \frac{1}{2} - \frac{1}{\vartheta}$$

for $|u|$ large, since $\tau + 1 > (\tau - 1)(q - 1)$. □

3. COERCIVE ELECTRIC POTENTIALS

In this section we always deal with a *fixed* value of $\lambda > 0$. The lack of compactness of the variational problem associated to (1.1) will be overcome by the following assumption on the potential V .

Definition 3.1. We say that $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a coercive electric potential if

$$(V1) \quad \text{ess inf}_{x \in \mathbb{R}^N} V(x) > 0;$$

$$(V2) \quad \lim_{|y| \rightarrow +\infty} \int_{B(y,1)} \frac{dx}{V(x)} = 0, \text{ where } B(y,1) = \{x \in \mathbb{R}^N \mid |y-x| < 1\}.$$

The term *coercive* has been used because the usual coercivity condition

$$\lim_{|x| \rightarrow +\infty} V(x) = +\infty$$

immediately implies (V2).

Remark 3.2. Of course the choice of $B(y,1)$ is fairly arbitrary: any ball of fixed radius $r > 0$ would do the same job.

Define the weighted Sobolev space

$$H = \left\{ u \in L^{\alpha,2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x) |u(x)|^2 dx < +\infty \right\}$$

equipped with the norm

$$(3.1) \quad \|u\|_H^2 = \int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} u \right|^2 dx + \int_{\mathbb{R}^N} V(x) |u(x)|^2 dx.$$

Since the norm $u \mapsto \int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} u \right|^2 dx$ already contains the L^2 norm, we can allow the inequality $V > 0$ to be true up to a subset of zero Lebesgue measure. In particular V may vanish on a curve, but not on an open set. Furthermore, equation (3.1) defines a norm even if V is negative: more precisely, it is enough to assume that

$$V(x) + \inf_{u \in L^{\alpha,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} u \right|^2 dx}{\int_{\mathbb{R}^N} |u|^2 dx} > 0$$

for every $x \in \mathbb{R}^N$.

Definition 3.3. We say that $u \in L^{\alpha,2}(\mathbb{R}^N)$ is a weak solution to (1.1) if

$$\begin{aligned} \int_{\mathbb{R}^N} (I - \Delta)^{\alpha/2} u (I - \Delta)^{\alpha/2} v dx + \int_{\mathbb{R}^N} \lambda V(x) u(x) v(x) dx \\ = \int_{\mathbb{R}^N} f(x, u(x)) v(x) dx + \int_{\mathbb{R}^N} \xi(x) |u|^{p-2} u v dx \end{aligned}$$

for all $v \in L^{\alpha,2}(\mathbb{R}^N)$, or, equivalently,

$$\begin{aligned} \int_{\mathbb{R}^N} (1 + |\xi|^2)^\alpha \mathcal{F}u(\xi) \mathcal{F}v(\xi) d\xi + \int_{\mathbb{R}^N} \lambda V(x) u(x) v(x) dx \\ = \int_{\mathbb{R}^N} f(x, u(x)) v(x) dx + \int_{\mathbb{R}^N} \xi(x) |u|^{p-2} uv dx. \end{aligned}$$

Remark 3.4. For a general measurable subset Ω of \mathbb{R}^N , the Bessel space $L^{\alpha,2}(\Omega)$ is defined as the set of restrictions to Ω of functions from $L^{\alpha,2}(\mathbb{R}^N)$. This will be useful in the following Proposition.

Proposition 3.5. *If V is a compact electric potential and $2 \leq q < 2_\alpha^*$, then the space H is compactly embedded into $L^q(\mathbb{R}^N)$.*

Proof. In the proof we will discard the set where V vanishes, since it has zero measure. We follow closely [6]. Let $\{u_n\}_n$ be a sequence from H such that $u_n \rightharpoonup 0$ as $n \rightarrow +\infty$. For some $M > 0$, we have

$$\int_{\mathbb{R}^N} V(x) |u(x)|^2 dx \leq M \quad \text{for all } n \in \mathbb{N}.$$

We define

$$\begin{aligned} \Theta_m &= \left\{ A \subset \mathbb{R}^N \mid A \text{ is measurable and } \lim_{|x| \rightarrow +\infty} \mathcal{L}^N(A \cap B(x, 1)) = 0 \right\} \\ \Theta_0 &= \{ \Omega \in \Theta_m \mid \Omega \text{ is open} \}. \end{aligned}$$

It follows from assumption (V1) that H embeds continuously into $L^{\alpha,2}(\mathbb{R}^N)$, and therefore the restriction of u_n to Ω converges weakly to zero in $L^{\alpha,2}(\Omega)$ for any $\Omega \in \Theta_0$. We can refer to Theorem 2.4 of [7] and conclude that $(u_n)|_\Omega \rightarrow 0$ strongly in $L^2(\Omega)$ for every $\Omega \in \Theta_0$.

Now pick any $\varepsilon > 0$. We compute for all $n \in \mathbb{N}$:

$$\|u_n\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \frac{1}{V(x)} |u_n(x)|^2 V(x) dx \leq M\varepsilon + \int_{V^{1/\varepsilon}} |u_n(x)|^2 dx,$$

where

$$V^{1/\varepsilon} = \{x \in \mathbb{R}^N \mid V(x) < 1/\varepsilon\}.$$

It follows from assumption (V2) that $V^{1/\varepsilon} \in \Theta_m$. We claim that there exists an open set $\Omega_\varepsilon \in \Theta_0$ such that $V^{1/\varepsilon} \subset \Omega_\varepsilon$. If this is the case, we recall that $(u_n)|_{\Omega_\varepsilon}$ converges to zero strongly in $L^2(\Omega_\varepsilon)$ and conclude easily that

$$\limsup_{n \rightarrow +\infty} \|u_n\|_{L^2(\mathbb{R}^N)}^2 \leq M\varepsilon + \int_{\Omega_\varepsilon} |u_n(x)|^2 dx \leq (M+1)\varepsilon.$$

To prove the claim, we introduce a countable family $\{\mathcal{O}_k\}_k$ of open sets such that $\mathcal{O}_1 = B(0, 1)$, $V^{1/\varepsilon} \cap (\overline{B(0, k+1)} \setminus B(0, k)) \subset \mathcal{O}_{k+1}$ and

$$\mathcal{L}^N \left(\mathcal{O}_{k+1} \setminus \left(V^{1/\varepsilon} \cap (\overline{B(0, k+1)} \setminus B(0, k)) \right) \right) < \frac{1}{k}.$$

We put $\Omega_\varepsilon = \bigcup_{k=1}^\infty \mathcal{O}_k$. Now,

$$\mathcal{L}^N(B(x, 1) \cap \Omega_\varepsilon) \leq \mathcal{L}^N(B(x, 1) \cap V^{1/\varepsilon}) + \mathcal{L}^N((\Omega_\varepsilon \setminus V^{1/\varepsilon}) \cap B(x, 1)).$$

We need to check that $\lim_{|x| \rightarrow +\infty} \mathcal{L}^N((\Omega_\varepsilon \setminus V^{1/\varepsilon}) \cap B(x, 1)) = 0$. Let $x \in \mathbb{R}^N$ and let n_1 be the largest integer such that $n_1 \leq |x| - 1$. We deduce from the properties of $\{\mathcal{O}_k\}_k$ that

$$\mathcal{L}^N((\Omega_\varepsilon \setminus V^{1/\varepsilon}) \cap B(x, 1)) \leq \mathcal{L}^N \left(\bigcup_{k=n_1+1}^{n_1+3} \mathcal{O}_k \setminus V^{1/\varepsilon} \right) < \frac{3}{n_1} < \frac{3}{|x| - 2}.$$

The claim is proved.

So far we have shown that H is compactly embedded into $L^2(\mathbb{R}^N)$. If $q \in [2, 2_\alpha^*)$, we recall that H is continuously embedded into $L^{2_\alpha^*}(\mathbb{R}^N)$ and the compactness of the embedding into $L^q(\mathbb{R}^N)$ follows from standard interpolation inequalities in Lebesgue spaces. \square

Remark 3.6. We notice that V is a coercive electric potential if, and only if,

$$\lim_{|x| \rightarrow +\infty} \mathcal{L}^N(B(x, 1) \cap V^b) = 0$$

for every $b \in \mathbb{R}$.

Theorem 3.7. *Assume that (f1), (f2), (f3), (V1) and (V2) hold. For every $\lambda > 0$ there exists $\mu_0 > 0$ such that for every $\mu \in (0, \mu_0)$, there exists at least a nontrivial solution to (1.1).*

We will prove Theorem 3.7 by variational methods. First of all, we associate to equation (1.1) the Euler functional $\Phi: H \rightarrow \mathbb{R}$ defined by

$$(3.2) \quad \Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} u \right|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} V(x) |u(x)|^2 dx - \Psi(u),$$

where

$$\Psi(u) = \int_{\mathbb{R}^N} F(u(x)) dx + \frac{\mu}{p} \int_{\mathbb{R}^N} \xi(x) |u(x)|^p dx.$$

It is easy to check that Φ is continuously differentiable on H under our assumptions. Moreover,

$$\begin{aligned} D\Phi(u)[v] &= \int_{\mathbb{R}^N} (I - \Delta)^{\alpha/2} u (I - \Delta)^{\alpha/2} v dx + \int_{\mathbb{R}^N} \lambda V(x) u(x) v(x) dx \\ D\Psi(u)[v] &= \int_{\mathbb{R}^N} f(x, u(x)) v(x) dx + \mu \int_{\mathbb{R}^N} \xi(x) |u(x)|^{p-2} u(x) v(x) dx, \end{aligned}$$

so that weak solutions to (1.1) correspond to critical points of Φ via Definition 3.3.

We will check that Φ satisfies the geometric assumptions of the Mountain Pass Theorem, see [2].

Lemma 3.8. *Let us retain the assumption of Theorem 3.7.*

- (i) *There exist three positive constants μ_0 , ρ and η such that $\Phi(u) \geq \eta$ for all $u \in H$ with $\|u\|_H = \rho$ and all $\mu \in (0, \mu_0)$.*
- (ii) *Let $\rho > 0$ be the number constructed in step (i). There exists $e \in H$ such that $\|e\|_H > \rho$ and $\Phi(e) < 0$ for all $\mu \geq 0$.*

Proof. Let us prove (i). For any fixed $\varepsilon > 0$, assumptions (f1) and (f2) imply that there is a positive constant C_ε such that

$$(3.3) \quad |F(x, u)| \leq \frac{\varepsilon}{2} |u|^2 + \frac{C_\varepsilon}{q} |u|^q$$

for all $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$. Integrating and using the Sobolev inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^N} F(x, u) dx &\leq \frac{\varepsilon}{2} \int_{\mathbb{R}^N} |u(x)|^2 dx + \frac{C_\varepsilon}{q} \int_{\mathbb{R}^N} |u(x)|^q dx \\ &\leq C \left(\frac{\varepsilon}{2} \|u\|_H^2 + \frac{C_\varepsilon}{q} \|u\|_H^q \right). \end{aligned}$$

Therefore

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \|u\|_H^2 - \int_{\mathbb{R}^N} F(x, u(x)) dx - \frac{\mu}{p} \int_{\mathbb{R}^N} \xi(x) |u(x)|^p dx \\ &\geq \frac{1}{2} \|u\|_H^2 - C \left(\frac{\varepsilon}{2} \|u\|_H^2 + \frac{C_\varepsilon}{q} \|u\|_H^q \right) - \frac{\mu}{p} C \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} \|u\|_H^p \\ &= \|u\|_H^p \left(\frac{1}{2} (1 - C\varepsilon) \|u\|_H^{2-p} - \frac{CC_\varepsilon}{q} \|u\|_H^{q-p} - \frac{\mu}{p} C \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} \right). \end{aligned}$$

We select (for instance) $\varepsilon = \frac{1}{2C}$ and maximize the function

$$g(t) = \frac{1}{4} t^{2-p} - \frac{CC_\varepsilon}{q} t^{q-p}$$

for $t \geq 0$. It is an exercise to check that the maximum is attained at some $\rho > 0$ where $g(\rho) > 0$.

We conclude by selecting $\mu_0 > 0$ so small that $g(\rho) - \frac{\mu_0}{p} C \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} > 0$.

To prove (ii), we recall (3.3). By assumption (f3), for some constant $c > 0$ we have

$$F(x, u) \geq c (|u|^\vartheta - |u|^2)$$

for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$. Given $u \in H$ and $t > 0$, we compute

$$\begin{aligned} \Phi(tu) &= \frac{t^2}{2} \|u\|_H^2 - \int_{\mathbb{R}^N} F(x, tu(x)) dx - \frac{\mu}{p} t^p \int_{\mathbb{R}^N} \xi(x) |u(x)|^p dx \\ &\leq \frac{t^2}{2} \|u\|_H^2 - ct^\vartheta \int_{\mathbb{R}^N} |u(x)|^\vartheta dx + ct^2 \int_{\mathbb{R}^N} |u(x)|^2 dx - \frac{\mu}{p} t^p \int_{\mathbb{R}^N} \xi(x) |u(x)|^p dx. \end{aligned}$$

Recalling that $1 < p < 2$ and $\vartheta > 2$, we can let $t \rightarrow +\infty$ and deduce that $\Phi(tu) \rightarrow -\infty$. The conclusion is now immediate. \square

We can now prove the main result of this section.

Proof of Theorem 3.7. For $0 < \mu < \mu_0$, the functional Φ satisfies the geometric assumptions of the Mountain Pass Theorem. As a consequence, there exist a Palais-Smale sequence $\{u_n\}_n$ from H , i.e.

$$\Phi(u_n) \rightarrow c, \quad D\Phi(u_n) \rightarrow 0$$

as $n \rightarrow +\infty$, where

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi(\gamma(t)) \quad \text{and} \quad \Gamma = \{\gamma \in C([0,1], H) \mid \gamma(0) = 0, \gamma(1) = e\}.$$

We prove that $\{u_n\}_n$ is bounded. Indeed,

$$\begin{aligned} 1 + c + \|u_n\|_H &\geq \Phi(u_n) - \frac{1}{\vartheta} D\Phi(u_n)[u_n] \\ &= \left(\frac{1}{2} - \frac{1}{\vartheta}\right) \|u_n\|_H^2 + \int_{\mathbb{R}^N} \left(\frac{1}{\vartheta} u_n(x) f(x, u_n(x)) - F(x, u_n(x))\right) dx \\ &\quad + \left(\frac{1}{\vartheta} - \frac{1}{p}\right) \int_{\mathbb{R}^N} \mu \xi(x) |u_n(x)|^p dx. \end{aligned}$$

Since

$$\begin{aligned} \left(\frac{1}{p} - \frac{1}{\vartheta}\right) \mu \int_{\mathbb{R}^N} \xi(x) |u_n(x)|^p dx &\leq \left(\frac{1}{p} - \frac{1}{\vartheta}\right) \mu \left(\int_{\mathbb{R}^N} |\xi(x)|^{\frac{2}{2-p}} dx\right)^{\frac{2-p}{2}} \left(\int_{\mathbb{R}^N} |u_n(x)|^2 dx\right)^{\frac{p}{2}} \\ &= \left(\frac{1}{p} - \frac{1}{\vartheta}\right) \mu \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} \|u_n\|_{L^2(\mathbb{R}^N)}^p \leq C \left(\frac{1}{p} - \frac{1}{\vartheta}\right) \mu \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} \|u_n\|_H^p, \end{aligned}$$

we derive that

$$\begin{aligned} 1 + c + \|u_n\|_H + C \left(\frac{1}{p} - \frac{1}{\vartheta}\right) \mu \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} \|u_n\|_H^p \\ \geq \left(\frac{1}{2} - \frac{1}{\vartheta}\right) \|u_n\|_H^2 + \int_{\mathbb{R}^N} \left(\frac{1}{\vartheta} u_n(x) f(x, u_n(x)) - F(x, u_n(x))\right) dx \\ \geq \left(\frac{1}{2} - \frac{1}{\vartheta}\right) \|u_n\|_H^2. \end{aligned}$$

Since $1 < p < 2$, this inequality shows that $\{u_n\}_n$ is a bounded sequence in H . By Proposition 3.5, $\{u_n\}_n$ converges up to a subsequence (weakly in H and) strongly in $L^2(\mathbb{R}^N)$ and in $L^q(\mathbb{R}^N)$ to some limit u . Since

$$\int_{\mathbb{R}^N} \xi(x) |u_n(x) - u(x)|^p dx \leq \|\xi\|_{L^{2/(2-p)}(\mathbb{R}^N)} \|u_n - u\|_{L^2(\mathbb{R}^N)}^p,$$

it follows immediately that $\{u_n\}_n$ is relatively compact in H , or, in other words, that Φ satisfies the Palais-Smale condition. Hence u is a critical point of Φ , namely a weak solution to (1.1). \square

We conclude this section with a regularity result. We skip its proof, since it is based on arguments that already appear in [19, 21, 26].

Theorem 3.9. *The solution $u \in H$ belongs to $C_b^\beta(\mathbb{R}^N)$ for every $\beta \in (0, 2\alpha)$. Here*

$$C_b^\beta(\mathbb{R}^N) = \left\{ u \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \mid \sup_{\substack{x, y \in \mathbb{R}^N \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|} < +\infty \right\} \quad \text{if } \beta < 1,$$

and

$$C_b^\beta(\mathbb{R}^N) = \left\{ u \in C^1(\mathbb{R}^N) \mid u \in L^\infty(\mathbb{R}^N), \nabla u \in L^\infty(\mathbb{R}^N) \cap C_b^{\beta-1}(\mathbb{R}^N) \right\} \quad \text{if } 1 < \beta < 2.$$

4. SOLUTIONS FOR LARGE VALUES OF λ AND SMALL VALUES OF μ

In this section we solve equation (1.1) under weaker conditions on the electric potential V . As we have seen in the previous section, the compactness of V yields the validity of the Palais-Smale condition almost for free. We show that we can relax the assumptions on V , provided that the parameter λ is sufficiently large.

Precisely, we assume the following:

(V3) $V \geq 0$ on \mathbb{R}^N ;

(V4) for some $b > 0$, the Lebesgue measure of the set $V^b = \{x \in \mathbb{R}^N \mid V(x) < b\}$ is finite;

(V5) the set¹ $\Omega = (V^{-1}(\{0\}))^\circ$ is nonempty and has a smooth boundary. Furthermore, $\overline{\Omega} = V^{-1}(\{0\})$.

Theorem 4.1. *Assume that (V3), (V4), (V5) and (f1), (f2), (f3) are satisfied. There exist two constants $\lambda_0 > 0$ and $\mu_0 > 0$ such that for every $\lambda > \lambda_0$ and every $0 < \mu < \mu_0$ equation (1.1) possesses at least two nontrivial solutions.*

Again we will prove this result by means of variational methods. Since λ is no longer fixed, we will use the notation Φ_λ for the Euler functional (3.2).

We define the space

$$\mathcal{H} = \left\{ u \in L^{\alpha, 2}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} V(x) |u(x)|^2 dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} \right|^2 dx + \int_{\mathbb{R}^N} V(x) |u(x)|^2 dx.$$

¹The notation A° is used to denote the interior of a set A .

For technical reasons, we will need to work with the norm

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} \right|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) |u(x)|^2 dx,$$

and we will write \mathcal{H}_λ to denote the space \mathcal{H} endowed with the norm $\|\cdot\|_\lambda$.

Lemma 4.2. *There exist two constants $\gamma_0 > 0$ and $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$ there results*

$$\|u\|_{L^{\alpha,2}(\mathbb{R}^N)} \leq \gamma_0 \|u\|_\lambda$$

for every $u \in \mathcal{H}_\lambda$.

Proof. Indeed, we get from the Sobolev embedding theorem

$$\begin{aligned} \int_{\mathbb{R}^N} |u(x)|^2 dx &= \int_{V^b} |u(x)|^2 dx + \int_{\mathbb{R}^N \setminus V^b} |u(x)|^2 dx \\ &\leq \left(\mathcal{L}^N(V^b) \right)^{\frac{2\alpha}{N}} \left(\int_{\mathbb{R}^N} |u(x)|^{2^*_\alpha} dx \right)^{\frac{N-2\alpha}{2}} + \int_{\mathbb{R}^N \setminus V^b} |u(x)|^2 dx \\ &\leq \left(\mathcal{L}^N(V^b) \right)^{\frac{2\alpha}{N}} \left(\int_{\mathbb{R}^N} |u(x)|^{2^*_\alpha} dx \right)^{\frac{N-2\alpha}{2}} + \frac{1}{\lambda b} \int_{\mathbb{R}^N \setminus V^b} \lambda V(x) |u(x)|^2 dx \\ &\leq C \left(\mathcal{L}^N(V^b) \right)^{\frac{2\alpha}{N}} \int_{\mathbb{R}^N} |(I - \Delta)u|^2 dx + \frac{1}{\lambda b} \int_{\mathbb{R}^N \setminus V^b} \lambda V(x) |u(x)|^2 dx, \end{aligned}$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} u \right|^2 dx &\leq \int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} u \right|^2 dx + \int_{\mathbb{R}^N} |u(x)|^2 dx \\ &\leq \left(1 + C \left(\mathcal{L}^N(V^b) \right)^{\frac{2\alpha}{N}} \right) \left(\int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} u \right|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) |u(x)|^2 dx \right) \\ &= \gamma_0 \left(\int_{\mathbb{R}^N} \left| (I - \Delta)^{\alpha/2} u \right|^2 dx + \int_{\mathbb{R}^N} \lambda V(x) |u(x)|^2 dx \right) \end{aligned}$$

whenever

$$\lambda \geq \lambda_0 = \frac{1}{b} \frac{1}{1 + C \left(\mathcal{L}^N(V^b) \right)^{\frac{2\alpha}{N}}}.$$

□

Corollary 4.3. *For all $s \in [2, 2^*_\alpha)$, there exists a constant $\gamma_s > 0$ such that*

$$\|u\|_{L^s(\mathbb{R}^N)} \leq \gamma_s \|u\|_{L^{\alpha,2}(\mathbb{R}^N)} \leq \gamma_0 \gamma_s \|u\|_\lambda$$

for every $u \in \mathcal{H}_\lambda$.

Proof. It suffices to combine the Sobolev embedding theorem with Lemma 4.2. □

The mountain-pass geometry of $\Phi_\lambda : \mathcal{H}_\lambda \rightarrow \mathbb{R}$ is ensured by Lemma 3.8. On the contrary, the Palais-Smale condition is now harder to prove, since no *coerciveness* assumption on the electric potential has been made.

Lemma 4.4. *Suppose that $u_n \rightharpoonup u_0$ in \mathcal{H}_λ as $n \rightarrow +\infty$. Then, up to a subsequence,*

$$(4.1) \quad \Phi_\lambda(u_n) = \Phi_\lambda(u_n - u_0) + \Phi_\lambda(u_0) + o(1)$$

and

$$(4.2) \quad D\Phi_\lambda(u_n) = D\Phi_\lambda(u_n - u_0) + D\Phi_\lambda(u_0) + o(1)$$

as $n \rightarrow +\infty$. In particular, if $\{u_n\}_n$ is a Palais-Smale sequence at level d , then

$$(4.3) \quad \Phi_\lambda(u_n - u_0) = d - \Phi_\lambda(u_0) + o(1), \quad D\Phi_\lambda(u_n - u_0) = o(1)$$

as $n \rightarrow +\infty$, up to a subsequence.

Proof. From the weak convergence assumption on u_n it follows that $\|u_n\|_\lambda^2 = \|u_n - u_0\|_\lambda^2 + \|u_0\|_\lambda^2 + o(1)$. To prove (4.1) and (4.2) it will be enough to check that as $n \rightarrow +\infty$

$$(4.4) \quad \int_{\mathbb{R}^N} (F(x, u_n(x)) - F(x, u_n(x) - u_0(x)) - F(x, u_0(x))) dx = o(1)$$

$$(4.5) \quad \int_{\mathbb{R}^N} \xi(x) (|u_n(x)|^p - |u_n(x) - u_0(x)|^p - |u_0(x)|^p) dx = o(1)$$

$$(4.6) \quad \int_{\mathbb{R}^N} (f(x, u_n(x)) - f(x, u_n(x) - u_0(x)) - f(x, u_0(x))) \phi(x) dx = o(1)$$

and

$$(4.7) \quad \int_{\mathbb{R}^N} \xi(x) (|u_n(x)|^{p-2} u_n(x) - |u_n(x) - u_0(x)|^{p-2} (u_n(x) - u_0(x)) - |u_0(x)|^{p-2} u_0(x)) \phi(x) dx = o(1)$$

for all $\phi \in \mathcal{H}_\lambda$. To prove (4.4) we follow the spirit of an idea due to Brezis and Lieb ([8]). We define $\delta_n = u_n - u_0$ so that $\delta_n \rightarrow 0$ in $L_{\text{loc}}^2(\mathbb{R}^N)$ and observe that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$(4.8) \quad |f(x, u)| \leq \varepsilon |u| + C_\varepsilon |u|^{q-1}$$

$$(4.9) \quad |F(x, u)| \leq \int_0^1 |f(x, tu)| |u| dt \leq \varepsilon |u|^2 + C_\varepsilon |u|^q$$

for every $(x, u) \in \mathbb{R}^N \times \mathbb{R}^N$. Hence

$$\begin{aligned} |F(x, \delta_n + u_0) - F(x, \delta_n)| &\leq \int_0^1 |f(x, \delta_n + \zeta u_0)| |u_0| d\zeta \\ &\leq \int_0^1 (\varepsilon |\delta_n + \zeta u_0| |u_0| + C_\varepsilon |\delta_n + \zeta u_0|^{q-1} |u_0|) d\zeta \\ &\leq C (\varepsilon |\delta_n| |u_0| + \varepsilon |u_0|^2 + C_\varepsilon |\delta_n|^{q-1} |u_0| + C_\varepsilon |u_0|^q), \end{aligned}$$

and the Young inequality for numbers implies that

$$|F(x, \delta_n + u_0) - F(x, \delta_n)| \leq C (\varepsilon |\delta_n|^2 + \varepsilon |u_0|^2 + C_\varepsilon |\delta_n|^q + C_\varepsilon |u_0|^q).$$

Using (4.9) we find similarly

$$|F(x, \delta_n + u_0) - F(x, \delta_n) - F(x, u_0)| \leq C (\varepsilon |\delta_n|^2 + \varepsilon |u_0|^2 + C_\varepsilon |\delta_n|^q + C_\varepsilon |u_0|^q).$$

We introduce

$$M_n(x) = (F(x, \delta_n + u_0) - F(x, \delta_n) - F(x, u_0) - \varepsilon |\delta_n|^2 - C_\varepsilon |\delta_n|^q) \vee 0,$$

where $a \vee b = \max\{a, b\}$. From the previous estimates it follows that $0 \leq M_n \leq \varepsilon |u_0|^2 + C_\varepsilon |u_0|^q \in L^1(\mathbb{R}^N)$ for every $n \in \mathbb{N}$. By dominated convergence, $M_n \rightarrow 0$ in $L^1(\mathbb{R}^N)$. Therefore

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^N} |F(x, \delta_n(x) + u_0(x)) - F(x, \delta_n(x)) - F(x, u_0(x))| dx \\ \leq C \varepsilon \limsup_{n \rightarrow +\infty} \left(\|\delta_n\|_{L^2(\mathbb{R}^N)}^2 + \|\delta_n\|_{L^q(\mathbb{R}^N)}^q \right), \end{aligned}$$

and (4.4) follows. To prove (4.5) we simply recall that $\delta_n \rightarrow 0$ strongly in $L_{\text{loc}}^2(\mathbb{R}^N)$. With the aid of the Hölder inequality, for any measurable set A we can write

$$\int_A \xi(x) |\delta_n(x)|^p dx \leq \left(\int_A |\xi(x)|^{\frac{2}{2-p}} dx \right)^{\frac{2-p}{2}} \left(\int_A |\delta_n(x)|^2 dx \right)^{\frac{p}{2}}.$$

We take $A = \mathbb{R}^N \setminus B(0, \bar{R})$ for a sufficiently large radius \bar{R} in such a way that $\int_{\mathbb{R}^N \setminus B(0, \bar{R})} |\xi(x)|^{\frac{2}{2-p}} dx$ is as small as we like. The term $\int_{B(0, \bar{R})} |\delta_n(x)|^2 dx$ is small due to the strong local convergence. We have thus proved that

$$\int_{\mathbb{R}^N} \xi(x) |\delta_n(x)|^p dx = o(1).$$

Since

$$\left| \int_{\mathbb{R}^N} \xi(x) (|u_n(x)|^p - |u_0(x)|^p) dx \right| \leq \int_{\mathbb{R}^N} \xi(x) |\delta_n(x)|^p dx,$$

the proof of (4.5) is complete. Reasoning in a very similar way we can also check the validity of (4.6) and (4.7). The last part of the Lemma is standard, and we omit it. \square

Lemma 4.5. *Assume that (V3), (V4), (V5) and (f1), (f2), (f3) hold. For some $\Lambda > 0$, the functional Φ_λ satisfies the Palais-Smale condition for any $\lambda \geq \Lambda$.*

Proof. We follow some ideas of [16]. As in the proof of Theorem 3.7, any Palais-Smale sequence $\{u_n\}_n$ for Φ_λ at level d is bounded. Up to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in \mathcal{H}_λ and $u_n \rightarrow u_0$ strongly in $L^r_{\text{loc}}(\mathbb{R}^N)$ for every $r \in [2, 2_\alpha^*)$. Writing again $\delta_n = u_n - u_0$, assumption (V4) implies that

$$(4.10) \quad \int_{\mathbb{R}^N} |\delta_n(x)|^2 dx \leq \frac{1}{\lambda b} \int_{\mathbb{R}^N \setminus V^b} \lambda V(x) |\delta_n(x)|^2 dx + \int_{V^b} |\delta_n(x)|^2 dx \leq \frac{1}{\lambda b} \|\delta_n\|_\lambda^2 + o(1).$$

and remark that

$$\begin{aligned} \int_{\mathbb{R}^N} \mathcal{F}(x, \delta_n(x)) dx &= \Phi_\lambda(\delta_n) - \frac{1}{2} D\Phi_\lambda(\delta_n)[\delta_n] - \left(\frac{1}{2} - \frac{1}{p}\right) \mu \int_{\mathbb{R}^N} \xi(x) |\delta_n(x)|^p dx \\ &= d - \Phi_\lambda(u_0) + o(1) \end{aligned}$$

by (4.3). Let

$$N_0 = \sup_{n \in \mathbb{N}} \left| \int_{\mathbb{R}^N} \mathcal{F}(x, \delta_n(x)) dx \right|, \quad \sigma = \frac{2\tau}{\tau - 1} \in (2, 2_\alpha^*).$$

From the Hölder inequality and Lemma 2.3,²

$$\begin{aligned} (4.11) \quad \int_{|\delta_n| \geq R} f(x, \delta_n(x)) \delta_n(x) dx &\leq \left(\int_{|\delta_n| \geq R} \left| \frac{f(x, \delta_n(x))}{\delta_n(x)} \right|^\tau dx \right)^{1/\tau} \left(\int_{|\delta_n| \geq R} |\delta_n(x)|^\sigma dx \right)^{2/\sigma} \\ &\leq \left(\int_{|\delta_n| \geq R} \mathcal{F}(x, \delta_n(x)) dx \right)^{1/\tau} \|\delta_n\|_{L^\sigma(\mathbb{R}^N)}^2 \leq N_0^{1/\tau} \|\delta_n\|_{L^\sigma(\mathbb{R}^N)}^2. \end{aligned}$$

We want to estimate the last norm of δ_n in terms of the norm in \mathcal{H}_λ . To do this, we pick $v \in (\sigma, 2_\alpha^*)$ and interpolate:

$$\begin{aligned} \|\delta_n\|_{L^\sigma(\mathbb{R}^N)}^\sigma &\leq \|\delta_n\|_{L^2(\mathbb{R}^N)}^{\frac{2(v-\sigma)}{v-2}} \|\delta_n\|_{L^v(\mathbb{R}^N)}^{\frac{v(\sigma-2)}{v-2}} \leq \left(\frac{1}{\lambda b} \right)^{\frac{v-\sigma}{v-2}} \|\delta_n\|_\lambda^{\frac{2(v-\sigma)}{v-2}} (\mathcal{V}_\mathcal{V} \|\delta_n\|_\lambda)^{\frac{v-\sigma}{v-2}} + o(1) \\ &\leq (\mathcal{V}_\mathcal{V})^{\frac{v-\sigma}{v-2}} \left(\frac{1}{\lambda b} \right)^{\frac{v-\sigma}{v-2}} \|\delta_n\|_\lambda^\sigma + o(1), \end{aligned}$$

where we have used (4.10). Going back to (4.11), for a suitable positive constant C ,

$$(4.12) \quad \int_{|\delta_n| \geq R} f(x, \delta_n(x)) \delta_n(x) dx \leq \left(\frac{C}{\lambda b} \right)^{\frac{2(v-\sigma)}{\sigma(v-2)}} \|\delta_n\|_\lambda^2 + o(1).$$

On the other hand,

$$(4.13) \quad \int_{|\delta_n| \leq R} f(x, \delta_n(x)) \delta_n(x) dx \leq \int_{|\delta_n| \leq R} (\varepsilon + C_\varepsilon R^{q-2}) |\delta_n(x)|^2 dx \leq \frac{C_\varepsilon R^{q-2}}{\lambda b} \|\delta_n\|_\lambda^2 + o(1).$$

² $R > 0$ is the number constructed in Lemma 2.3

Combining now (4.12) with (4.13) we obtain

$$\begin{aligned} o(1) &= D\Phi_\lambda(\delta_n)[\delta_n] = \|\delta_n\|_\lambda^2 - \int_{\mathbb{R}^N} f(x, \delta_n(x)) \delta_n(x) dx - \mu \int_{\mathbb{R}^N} \xi(x) |\delta_n(x)|^p dx \\ &\geq \left(1 - C \left(\frac{1}{\lambda b} - \left(\frac{1}{\lambda b} \right)^{\frac{2(v-\sigma)}{\sigma(v-2)}} \right) \right) \|\delta_n\|_\lambda^2 + o(1). \end{aligned}$$

It now suffices to choose $\Lambda > 0$ so large that the last bracket is strictly positive for every $\lambda \geq \Lambda$, and we deduce that $\delta_n = o(1)$ as $n \rightarrow +\infty$. \square

We can now prove the main result of this section.

Proof of Theorem 4.1. First of all, we fix μ_0 such that Φ_λ has the mountain-pass geometry, see Lemma 3.8. Now we can introduce the value

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \sup_{t \in [0,1]} \Phi_\lambda(\gamma(t)),$$

where $\Gamma_\lambda = \{\gamma \in C([0,1], \mathcal{H}_\lambda) \mid \gamma(0) = 0, \gamma(1) = e\}$. By Lemma 4.5 there exists a number $\lambda_0 > 0$ such that Φ_λ satisfies the Palais-Smale condition at level c_λ for any $\lambda \geq \lambda_0$. Hence a first solution to equation (1.1) arises as a mountain-pass point at level $c_\lambda > 0$.

To construct the second solution, we remark that there always exists a function $\phi_0 \in \mathcal{H}_\lambda$ such that $\int_{\mathbb{R}^N} \xi(x) |\phi_0(x)|^p dx > 0$. On the straight half-line $t \mapsto t\phi_0$, we have

$$\Phi_\lambda(t\phi_0) = \frac{t^2}{2} \|\phi_0\|_\lambda^2 - \int_{\mathbb{R}^N} F(x, t\phi_0(x)) dx - \frac{\mu t^p}{p} \int_{\mathbb{R}^N} \xi(x) |\phi_0(x)|^p dx.$$

Since F is non-negative and $1 < p < 2$, there exists t_0 close to zero (without loss of generality we assume that $t < \rho$) such that $\Phi_\lambda(t_0\phi_0) < 0$. On the contrary, we already know that $\Phi_\lambda(u) > 0$ if $\|u\|_\lambda = \rho$. For

$$m_\lambda = \inf \left\{ \Phi_\lambda(u) \mid u \in \overline{B(0, \rho)} \right\} < 0$$

there exists a sequence $\{v_n\}_n$ in \mathcal{H}_λ such that $\Phi_\lambda(v_n) \rightarrow m_\lambda < 0$. From the previous discussion it is not restrictive to assume that v_n is far from the boundary of $\overline{B(0, \rho)}$. Hence the Ekeland Variational Principle implies that we may assume without loss of generality that $D\Phi_\lambda(v_n) = o(1)$ as $n \rightarrow +\infty$.

Taking as usual λ large and μ small enough, the Palais-Smale condition is satisfied at level m_λ , so that we may assume $v_n \rightarrow v_0$ strongly in \mathcal{H}_λ . Then v_0 is another solution of (1.1) at level $m_\lambda < 0$, and proof is complete. \square

Remark 4.6. Theorem 3.9 applies to u and v_0 as well, so that our solutions are more regular.

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